5 $C^{1,\alpha}-$blowup solution to inviscid flow

We recall the vorticity-stream formulation of the 3D Euler flow:

$$\begin{cases} \frac{1}{2}\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ -\Delta \psi = \omega, \\ u = \nabla \times \psi. \end{cases}$$

5.1 Formulation

Particularly, for the axisymmetric flow without swirl, the formulation is transformed as following under cylindrical system for in variables (r, x_3, t):[11]

$$\begin{cases} -\frac{1}{2}\partial_t \omega + u^r \partial_r \omega + u^3 \partial_3 \omega = \frac{u^r}{r}\omega, \\ \partial_r^2 \psi + \frac{1}{r}\partial_r \psi - \frac{1}{r^2}\psi + \partial_3 \psi = -\omega, \\ (u^r, u^3) = (\partial_3 \psi, -\partial_r \psi - \frac{1}{r}\psi). \end{cases}$$

Moreover, if we set the $\alpha-$related spherical coordinate:

$$\rho = \sqrt{r^2 + x_3^2}, \tan\theta = \frac{x_3}{r}, R = \rho^\alpha,$$

and let $\omega(r, x_3, t) = \Omega(R, \theta, t), \psi(r, x_3, t) = \rho^2 \Psi(R, \theta, t)$, then the spherical form is

$$\begin{cases} \frac{1}{2}\partial_t \Omega + U(\Psi)\partial_\theta \Omega + V(\Psi)\alpha D_R \Omega = R(\Psi)\Omega, \\ L(\Psi) = -\Omega, \end{cases} \tag{5.1}$$

where the linear operators involved are defined as

$$\begin{aligned} &U \coloneqq -3\,\mathrm{Id} - \alpha D_R, V \coloneqq \partial_\theta - \tan\theta, \\ &R \coloneqq \frac{1}{\cos\theta}\left(2\sin\theta + \alpha\sin\theta D_z + \cos\theta\partial_\theta\right), \\ &L \coloneqq L_R + L_\theta \coloneqq (\alpha^2 D_R^2 + \alpha(5+\alpha)D_R) + (\partial_\theta + \partial_\theta(\tan\theta\cdot) - 6\,\mathrm{Id}). \end{aligned}$$

It is noticeable that $\sin 2\theta$ is in the kernel of L_θ and $\sin\theta\cos^2\theta$ is in the kernel of L_θ^*, i.e.

$$L_\theta(2\theta) = 0 \text{ and } \left(L_\theta f, \sin\theta\cos^2\theta\right)_{L_\theta^2} = 0, \forall f \in L_\theta^2([0, \frac{\pi}{2}]).$$

[11] Indeed for these equations, ω, ψ correspond to the angular component in the cylindrical system of the vorticity and stream respectively.

Following we will construct angular weights according to these facts. Now let $z = \frac{R}{(1-(1+\mu))^{1+\lambda}}$ and consider the self-similar ansatz as:

$$\Omega(R, \theta, t) = \frac{1}{1-(1+\mu)t} F(z, \theta), \Psi(R, \theta, t) = \frac{1}{1-(1+\mu)t} \Phi(z, \theta).$$

Substitute them into the spherical form (5.1), then we obtain the profile equations:

$$\begin{cases} (1+\mu)F + (1+\mu)(1+\lambda)D_z F + 2U(\Phi)\partial_\theta F + 2\alpha V(\Phi) D_z F = 2R(\Phi)F, \\ \alpha^2 D_z \Phi + \alpha(5+\alpha)D_z\Phi + \partial_\theta^2 \Phi + \partial_\theta(\tan\theta\Phi) - 6\Phi = -F. \end{cases} \tag{5.2}$$

5.2 Weighted Sobolev spaces

Following we will introduce some weighted spaces which suit our topic. Define the radial weight, angular weight and weak angular weight respectively as

$$w_z(z) = \frac{(1+z)^2}{z^2}, w_\theta = (\sin\theta\cos^2\theta)^{-\frac{\gamma}{2}}, v_\theta = (\sin\theta\cos^2\theta)^{-\frac{\eta}{2}}$$

with $\gamma = 1 + \frac{\alpha}{10}$ and $\eta = \frac{99}{100}$. Now we define $\mathcal{H}^k([0,\infty) \times [0, \frac{\pi}{2}])$ and $\mathcal{W}^{l,\infty}([0,\infty) \times [0, \frac{\pi}{2}])$ as closure of $C_c^\infty([0,\infty) \times [0, \frac{\pi}{2}])$ in the following norms respectively:

$$\|f\|_{\mathcal{H}^k}^2 := \sum_{i \le k} \left\| D_z^i f w_z v_\theta \right\|_{L^2}^2 + \sum_{i+j \le k, j>0} \left\| D_z^i D_\theta^j f w_z w_\theta \right\|_{L^2}^2,$$

$$\|f\|_{W^{l,\infty}} := \sum_{i \le l} \left\| \tilde{D}_z^i f \right\|_{L^\infty} + \sum_{i+j \le l, j \ge 0} \left\| \tilde{D}_z^i D_\theta^j f \frac{\sin^{-\frac{\alpha}{5}} 2\theta}{\alpha + \sin 2\theta} \right\|_{L^\infty},$$

where $D_z = z\partial_z, \tilde{D}_z = (z+1)\partial_z$ and $D_\theta = \sin(2\theta)\partial_\theta$. We will show that

$$\Phi_s = \frac{1}{4\alpha} \sin 2\theta L_{12} F$$

is the main singular term of Φ during elliptic estimate in weighted spaces(see Theorem 5.5 and its remark), where the operator L_{12} is defined by

$$L_{12} f(z) := \int_z^\infty \int_0^{\frac{\pi}{2}} \frac{f(z', \theta') K(\theta')}{z'} d\theta' dz', K(\theta) = 3\sin\theta\cos^2\theta.$$

Consequently, let $\hat{\Phi} = \Phi - \Phi_s$, then the vorticity profile of (5.2) can be written as

$$(1+\mu)F + (1+\mu)(1+\lambda)D_z F + \frac{1}{2\alpha} U(\sin 2\theta L_{12} F)\partial_\theta F + \frac{1}{2} V(\sin 2\theta L_{12} F) D_z F$$
$$- \frac{1}{2\alpha} R(\sin 2\theta L_{12} F) F = -2U(\hat{\Phi})\partial_\theta F - 2\alpha V(\hat{\Phi}) D_z F + 2R(\hat{\Phi}) F.$$

As the explicit form of U, V, R and Φ_s are already given above, we can calculate out

$$\begin{aligned}U(\sin 2\theta L_{12}F) =& -3\sin 2\theta L_{12}F + \alpha \sin 2\theta (F,K)_\theta,\\ V(\sin 2\theta L_{12}F) =& 2(\cos 2\theta - \sin^2\theta) L_{12}F,\\ R(\sin 2\theta L_{12}F) =& 2L_{12}F - 2\alpha\sin^2\theta (F,K)_\theta.\end{aligned}$$

Here L_{12}−related terms will be the main difficulties, so we preserve them in the left hand and write the equation as

$$F + D_zF - \frac{1}{\alpha}FL_{12}F - \left(\frac{3}{2\alpha}L_{12}FD_\theta F - (\cos 2\theta - \sin^2\theta)L_{12}FD_zF\right) = -\mu F - (\mu+\lambda+\mu\lambda)F + \mathcal{N},$$

where the remain term

$$\mathcal{N} = -\frac{1}{2\alpha}U(\hat{\Phi})\partial_\theta F - \frac{1}{2}V(\hat{\Phi})D_zF + \frac{1}{2\alpha}R(\hat{\Phi})F - (F,K)_\theta\left(\frac{1}{2}D_\theta F + (\sin^2\theta)F\right).$$

The first part (except the transport terms) is the fundamental model with explicit solution

$$F_*(z,\theta) = \alpha\frac{\Gamma(\theta)}{c^*}\frac{2z}{(1+z)^2},$$

where $c^* = \int_0^{\frac{\pi}{2}} K(\theta)\Gamma(\theta)d\theta$ and $\Gamma(\theta) = (\sin\theta\cos^2\theta)^{\frac{\alpha}{3}}$(see Theorem 5.2). So we'd like express $F = F_* + g$ and then

$$g + D_zg - \frac{1}{\alpha}(L_{12}F_*)g - \frac{1}{\alpha}F_*L_{12}g - \frac{3}{2\alpha}(L_{12}F_*)D_\theta g = -\mu F_* - (\mu+\lambda+\mu\lambda)D_zF_* + \mathcal{N}_* + \mathcal{N} + \mathcal{N}_0 \tag{5.3}$$

with

$$\begin{aligned}\mathcal{N}_0 =& \frac{1}{\alpha}gL_{12}g + \frac{3}{2\alpha}(L_{12}F_*)D_\theta F_* + \frac{3}{2\alpha}(L_{12}g)D_\theta F - (\cos 2\theta - \sin^2\theta)L_{12}FD_zF,\\ \mathcal{N}_* =& -\mu g - (\mu+\lambda+\mu\lambda)D_zg.\end{aligned}$$

Notice that

$$\begin{aligned}L_{12}F_* =& \alpha\int_z^\infty\int_0^{\frac{\pi}{2}}\frac{F(\theta)K(\theta)}{c^*}\frac{2}{(1+z')^2}d\theta dz'\\ =& \alpha\int_z^\infty\frac{2}{(1+z')^2}dz' = \frac{2\alpha}{1+z}.\end{aligned}$$

Consequently, the left hand side of equation (5.3) can be expressed explicitly as

$$\mathcal{L}_\Gamma g - \frac{3}{1+z}D_\theta g,$$

with the operator $\mathcal{L}_\Gamma$ defined by

$$\mathcal{L}_\Gamma f := \mathcal{L}f - \frac{\Gamma}{c^*}\frac{2z}{(1+z)^2}L_{12}f := f + D_zf - \frac{2}{1+z}f - \frac{\Gamma}{c^*}\frac{2z}{(1+z)^2}L_{12}f.$$

To estimate the term $\frac{3}{1+z}D_\theta g$, we give some observation first: Notice the right hand side of (5.3). We can calculate out that

$$-\mu F_* - (\mu+\lambda+\mu\lambda)D_z F_* = -\frac{2\mu\alpha\Gamma(\theta)}{c^*}\left(\left(1+\frac{\mu+\lambda+\mu\lambda}{\mu}\right)\frac{z}{(1+z)^2} - 2\frac{\mu+\lambda+\mu\lambda}{\mu}\frac{z^2}{(1+z)^3}\right).$$

The first term is eliminated if we set $\mu + (\mu+\lambda+\mu\lambda) = 0 \Longleftrightarrow \lambda = \frac{-2\mu}{\mu+1}$. In this case, we have

$$\mathcal{L}_\Gamma g - \frac{3}{1+z}D_\theta g = -2\alpha\mu\frac{\Gamma}{c^*}\frac{2z^2}{(1+z)^3} + \mathcal{N}_* + \mathcal{N} + \mathcal{N}_0.$$

Motivated by this argument, we introduce a projector $\mathbb{P}$ on $\mathcal{H}([0,\infty)\times[0,\pi/2])(\mathbb{P}^2 = \mathbb{P}$ since it holds $L_{12}(\mathbb{P}(f))(0) = 0$ for any f):[12]

$$\mathbb{P}(f)(z,\theta) = f(z,\theta) - \frac{\Gamma(\theta)}{c^*}\frac{2z^2}{(1+z)^3}L_{12}f(0).$$

Consequently, we get

$$\mathcal{L}_\Gamma^T g \coloneqq \mathcal{L}_\Gamma g - \mathbb{P}\left(\frac{3}{1+z}D_\theta g\right) = \frac{\Gamma(\theta)}{c^*}\frac{2z^2}{(1+z)^3}\left(L_{12}\left(\frac{3}{1+z}D_\theta g\right)(0) - 2\alpha\mu\right) + \mathcal{N}_* + \mathcal{N} + \mathcal{N}_0.$$

Moreover, we notice that $L_{12}g(0) \Longrightarrow L_{12}(\mathcal{N}_*)(0) = 0$. Under this condition, the equation can be transform as following form

$$\mathcal{L}_\Gamma^T g = \mathbb{P}(\mathcal{N}_* + \mathcal{N} + \mathcal{N}_0),$$

if we let

$$L_{12}\left(\frac{3}{1+z}D_\theta g\right)(0) - 2\alpha\mu = -L_{12}(\mathcal{N}+\mathcal{N}_0)(0) \Longleftrightarrow \mu = \frac{1}{2\alpha}L_{12}\left(\mathcal{N} + \mathcal{N}_0 + \frac{3}{1+z}D_\theta g\right)(0).$$

Finally, we conclude our discussion as the following theorem:

Theorem 5.1. *Suppose $F = F_* + g$ is a solution of system (5.2) with*

$$F_*(z,\theta) = \alpha\frac{\Gamma(\theta)}{c^*}\frac{2z}{(1+z)^2}. \tag{5.4}$$

Then g satisfies the following equation

$$\begin{aligned}
\mathcal{L}_\Gamma^T g =& \mathbb{P}(\mathcal{N}_* + \mathcal{N} + \mathcal{N}_0),\\
\mathcal{N}_* =& -\mu g - (\mu+\lambda+\mu\lambda)D_z g,\\
\mathcal{N} =& 2R(\hat{\Phi})F - 2U(\hat{\Phi})\partial_\theta F - 2\alpha V(\hat{\Phi})D_z F\\
&- (F,K)_\theta\left(\frac{1}{2}D_\theta F + (\sin^2\theta)F\right),\\
\mathcal{N}_0 =& \frac{1}{\alpha}gL_{12}g + \frac{3}{1+z}D_\theta F_* + \frac{3}{2\alpha}(L_{12}g)D_\theta F\\
&- (\cos 2\theta - \sin^2\theta)L_{12}F D_z F,
\end{aligned} \tag{5.5}$$

[12]Notice $\int_0^\infty \frac{2z}{(1+z)^3}dz = -\frac{2x+1}{(x+1)}|_0^\infty = 1$. Particularly, the image of the projector is the functions f which satisfies $L_{12}(f)(0) = 0$.

if we assume the coefficient relation and restriction condition as

$$\mu = \frac{1}{2\alpha} L_{12} \left(\mathcal{N} + \mathcal{N}_0 + \frac{3}{1+z} D_\theta g \right)(0), \lambda = -\frac{2\mu}{1+\mu} \text{ and } L_{12}(g) = 0. \tag{5.6}$$

In the further investigation, we will establish the coercivity of the transport operator $\mathcal{L}_\Gamma^T$ (see Theorem 5.4) and elliptic estimates of $\hat{\Phi}$ (see Theorem 5.5) in the weighted space. Applying these two, a priori estimate will be obtained for g. Then the existence follows from a compactness argument.

5.3 Fundamental model

The idea of seeking for a fundamental model is inspired by the former work of Elgindi, where he neglects the transport term and focuses on the vortex stretching. Respectively in our case, we eliminate the term $U(\Psi)\partial_\theta F, V(\Psi)D_z F$ and obtain

$$\begin{cases} \frac{1}{2}\partial_t \Omega = R(\Phi)\Omega, \\ L\Psi = -\Omega. \end{cases}$$

However, this model is not explicit enough that we can give out a precise solution. And the idea is analyze the singularity of Ψ and observe that

$$\Psi = -L^{-1}\Omega = \frac{1}{4\alpha} \sin 2\theta L_{12}\Omega + \text{low order terms}.$$

Substitute the singular part into stretching operator:

$$R(\Psi)\Omega = \frac{1}{2\alpha} L_{12}\Omega - \frac{1}{2}\sin^2\theta \, (\Omega, K)_\theta \, .$$

As the later term is of low order (eliminated by Cauchy-Schwartz). Our main concern is now degenerated as the following form:

$$\partial_t \Omega = \frac{1}{\alpha}\Omega L_{12}\Omega. \tag{5.7}$$

The following theorem gives a explicit self-similar solution for the equation (5.7).

Theorem 5.2. *The fundamental model (5.7) possesses a family of self-similar solution of the form*

$$\Omega(R,\theta,t) = \frac{1}{1-t} F_* \left(\frac{R}{1-t}, \theta \right) = \frac{1}{1-t} \alpha \frac{\Gamma(\theta)}{c^*} F_{*,r} \left(\frac{R}{1-t} \right),$$

where $F_{*,r} = \frac{2z}{(1+z)^2}$ *and* $c^* = \int_0^{\frac{\pi}{2}} K(\theta)\Gamma(\theta)d\theta$. *Here* $\Gamma(\theta)$ *is some undetermined function satisfies* $K\Gamma \in L^1_\theta$.

Remark 5.1. *Particularly,* F_* *satisfies the profile equation* $F_* + D_z F_* = \frac{1}{\alpha} F_* L_{12} F_*$ *for variable* $z = \frac{R}{1-t}$.

Proof. Check later. ☐

5.4 Transport coercivity

We define the following quantities and operators: Suppose the undetermined $z \in [0, \infty), \theta \in [0, \frac{\pi}{2}]$. And denote the following coefficients:

$$\alpha \in (0,1), \gamma = 1 + \frac{\alpha}{10}, \eta = \frac{99}{100}.$$

Now suppose $f(z,\theta)$ and angular kernels

$$K(\theta) = 3\sin\theta\cos\theta^2, \Gamma(\theta) = (\sin\theta\cos\theta^2)^{\frac{\alpha}{3}},$$

then we set operators:

$$\begin{aligned} L_{12}f(z) =& \int_z^\infty \int_0^{\frac{\pi}{2}} \frac{f(z',\theta')K(\theta')}{z'} d\theta' dz', \\ \mathcal{L}f(z,\theta) =& f + D_z f - \frac{2f}{1+z}, \\ \mathcal{L}_\Gamma f(z,\theta) =& \mathcal{L}f - \frac{\Gamma}{c^*} \cdot \frac{2z}{(1+z)^2} L_{12} f, \\ \mathcal{L}_\Gamma^T f(z,\theta) =& \mathcal{L}_\Gamma f - \mathbb{P}\left(\frac{3}{1+z} D_\theta f\right), \end{aligned}$$

where $D_z = z\partial_z, D_\theta = \sin(2\theta)\partial_\theta$,

$$c^* = \int_0^{\frac{\pi}{2}} K(\theta)\Gamma(\theta) d\theta,$$

and $\mathbb{P}$ is a projector defined by

$$\mathbb{P}f(z,\theta) = f - \frac{\Gamma}{c^*} \cdot \frac{2z^2}{(1+z)^3} L_{12} f(0).$$

There the image of $\mathbb{P}$ consist with functions $g = \mathbb{P}f$ satisfies $L_{12}(g)(0)$. Indeed, we can check that

$$L_{12}\left(\frac{\Gamma(\theta)}{c^*}\frac{2z^2}{(1+z)^3}\right)(0) = \int_0^\infty \frac{2z}{(1+z)^3} dz \int_0^{\frac{\pi}{2}} K(\theta)\Gamma(\theta)/c^* d\theta = 1,$$

and then

$$L_{12}g(0) = L_{12}(\mathbb{P}f)(0) = L_{12}f(0) - L_{12}f(0) = 0.$$

Moreover, we assume the following radial and angular weights

$$w_z = \frac{(1+z)^2}{z^2}, w_\theta = \sin(2\theta)^{-\frac{\gamma}{2}}, v_\theta = \sin(2\theta)^{-\frac{\eta}{2}}.$$

Lemma 5.1. *Some important relations:*

1. $L_{12} \circ \mathcal{L}_\Gamma = \mathcal{L} \circ L_{12}$;

2. $\mathcal{L} f w_z = f w_z + D_z(f w_z)$.

Proof. The first is directly from the Tricomi identity:

$$L_{12}(f L_{12} g + g L_{12} f) = L_{12} f L_{12} g.$$

The second is from the fact that w_z satisfies the following equation:

$$D_z w_z + \frac{1}{\alpha} L_{12} F_* w_z = 0.$$

And consequently,

$$\begin{aligned}\mathcal{L} f w_z =& f w_z + D_z f w_z - \frac{1}{\alpha}(L_{12} F_*) f w_z \\ =& f w_z + D_z(f w_z) - f D_z w - \frac{1}{\alpha}(L_{12} F_*) f w_z \\ =& f w_z + D_z(f w_z).\end{aligned}$$

□

Before the further discussions, we lists some facts related to the kernel elements:

Proposition 5.1. *1.* $c^* = 3\int_0^{\frac{\pi}{2}} (\sin\theta\cos^2\theta)^{1+\frac{\alpha}{3}} d\theta \in (\frac{5\pi}{24}, 1)$ *for* $\alpha \in (0,1)$;

2. $\left\| \frac{\Gamma}{c^*} - K \right\|_{L^2_\theta} \leq \frac{7}{10}$;

3. $|D_\theta \Gamma| \leq 2\alpha\Gamma$.

The following theorem tells the coercivity of above transport operators.

Theorem 5.3. *Suppose* $L_{12}, \mathcal{L}, \mathcal{L}_\Gamma, \mathcal{L}_\Gamma^T$ *are defined as above. Then we have*

$$\|L_{12} f w_z\|_{L^2} \leq 4\|f w_z\|_{L^2}, \tag{5.8}$$

$$(\mathcal{L}_\Gamma f w_z, f w_z)_{L^2} \geq \frac{1}{4}\|f w_z\|_{L^2}^2, \tag{5.9}$$

$$\left(\mathcal{L}_\Gamma^T f w_z, f w_z\right)_{L^2} \geq \frac{1}{5}\|f w_z\|_{L^2}^2 - 100\|D_\theta f w_z\|_{L^2}^2, \tag{5.10}$$

$$\left(D_\theta(\mathcal{L}_\Gamma^T f) w_z w_\theta, D_\theta f w_z w_\theta\right)_{L^2} \geq \left(\frac{1}{4} - \alpha\right)\|D_\theta f w_z w_\theta\|_{L^2}^2 - 10^7\alpha\|f w_z\|_{L^2}^2, \tag{5.11}$$

$$\left(\mathcal{L}_\Gamma^T f w_z v_\theta, f w_z v_\theta\right)_{L^2} \geq \frac{1}{5}\|f w_z w_\theta\|_{L^2}^2 - 10^5\left\|\frac{1}{z} L_{12} f\right\|_{L^2}^2, \tag{5.12}$$

$$\left(D_z(\mathcal{L}_\Gamma^T f) w_z v_\theta, D_z f w_z v_\theta\right)_{L^2} \geq \frac{1}{4}\|D_z f w_z v_\theta\|_{L^2}^2 - 10^8\|D_\theta f w_z v_z\|_{L^2}^2 - 10^8\|f w_z w_\theta\|_{L^2}^2, \tag{5.13}$$

Proof of (5.8). We notice that

$$w_z^2 = \frac{(1+z)^4}{z^4} = 1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4} \sim 1 + 6z^{-2} + z^{-4}.$$

Then it is enough to show that

$$\left\| z^{-\frac{k}{2}} L_{12} f \right\|_{L_z^2} \leq 4 \left\| z^{-\frac{k}{2}} f \right\|_{L_{z,\theta}^2}, \forall k \text{ is even}.$$

□

Proof of (5.11). We notice that

$$\mathcal{L}_\Gamma^T f = \mathcal{L} f - \frac{3}{1+z} D_\theta f - \frac{2}{c^*} \Gamma \left(\frac{z}{(1+z)^2} L_{12} f + \frac{z^2}{(1+z)^3} L_{12} \left(\frac{3}{1+z} D_\theta f \right)(0) \right).$$

We denote $Af(z)$ the bracketed term in the last term, then it comes

$$\begin{aligned}
&\left(D_\theta(\mathcal{L}_\Gamma^T f) w_z w_\theta, D_\theta f w_z w_\theta \right)_{L^2} \\
=& (\mathcal{L}(w_\theta D_\theta f) w_z, w_\theta D_\theta f w_z)_{L^2} - \frac{3}{2} \left(\sin(2\theta)^{-(1+\frac{\alpha}{10})}, \frac{w_z^2}{1+z} D_\theta((D_\theta f)^2) \right)_{L^2} \\
&- \frac{2}{c^*} \int_0^\infty \int_0^{\frac{\pi}{2}} D_\theta \Gamma \cdot Af \cdot D_\theta f \cdot w_z^2 w_\theta^2 \, d\theta dz \\
\geq& \frac{1}{4} \| D_\theta f w_z w_\theta \|_{L^2}^2 + \frac{3}{2} \left(\partial_\theta (\sin(2\theta)^{-\frac{\alpha}{10}}), \frac{w_z^2}{1+z} (D_\theta f)^2 \right)_{L^2} \\
&- \frac{2}{c^*} \| D_\theta \Gamma w_\theta \|_{L_\theta^2} \| Af w_z \|_{L_z^2} \| D_\theta f w_z w_\theta \|_{L^2} \\
\geq& \frac{1}{4} \| D_\theta f w_z w_\theta \|_{L^2}^2 - \frac{3}{2} \cdot \frac{\alpha}{10} \cdot 2 \left(\sin(2\theta)^{-(1+\frac{\alpha}{10})}, \frac{w_z^2}{1+z} (D_\theta f)^2 \right)_{L^2} \\
&- (4\pi\alpha)^{\frac{1}{2}} \cdot 39 \| f w_z \|_{L^2} \| D_\theta f w_z w_\theta \|_{L^2} \\
\geq& (\frac{1}{5} - \alpha) \| D_\theta f w_z w_\theta \|_{L^2}^2 - 10^7 \alpha \| f w_z \|_{L^2},
\end{aligned}$$

where we have apply the following estimates for $D_\theta \Gamma$ and Af: □

Proof of (5.12). □

With the above estimates, it is natural to define the weighted Sobolev space as following:

$$\| f \|_{\mathcal{H}^k}^2 = \sum_{i=0}^k \left\| D_z^i f w_z v_\theta \right\|_{L^2}^2 + \sum_{i+j \leq k, j \geq 0} \left\| D_z^i D_\theta^j f w_z w_\theta \right\|_{L^2}^2. \tag{5.14}$$

Then we claim the standard coercivity of L_Γ^T in $\mathcal{H}^k$:

Theorem 5.4. *Fix $\alpha \leq 10^{-14}$ and $k \in \mathbb{N}$. Then there exists C_k that for any $f \in \mathcal{H}^k$,*

$$\left(L_\Gamma^T f, f \right)_{\mathcal{H}^k} \geq C_k \| f \|_{\mathcal{H}}^k.$$

5.5 Elliptic estimate

We recall the profile equation of stream equation derived in Section **??**:

$$L_z\Phi + L_\theta\Phi = \alpha^2 D_z^2\Phi + \alpha(5+\alpha)D_z\Phi + \partial_\theta^2\Phi - \partial_\theta(\tan\theta\Phi) + 6\Phi = -F, \tag{5.15}$$

with Dirichlet boundary conditions

$$\Phi(z,0) = \Phi\left(z, \frac{\pi}{2}\right) = 0 \text{ and } \Phi(z,\theta) \to 0 \text{ as } z \to 0. \tag{5.16}$$

The main result of this section is the following $\mathcal{H}^k-$elliptic estimates:

Theorem 5.5. *Fix $k \geq 2$, then there eixsts $C_k > 0$ such that for any $\alpha \in [0, 1/4], \gamma \in [1, 5/4]$, if $F \in \mathcal{H}^k$ satisfies the following orthogonal condition*

$$F_\star(z) := \big(F(z,\theta), \sin\theta\cos^2\theta\big)_{L^2_\theta} \equiv 0, \tag{5.17}$$

then there exists a unique $\mathcal{H}^k-$solution Φ to (5.15)-(5.16) on $[0,\infty)\times[0,\pi/2]$, which satisfies

$$\alpha^2\big\|D_z^2\Phi\big\|_{\mathcal{H}^k} + \big\|\partial_\theta^2\Phi\big\|_{\mathcal{H}^k} \leq C_k\|F\|_{\mathcal{H}^k}. \tag{5.18}$$

Remark 5.2. *Notice that $\sin\theta\cos^2\theta$ is the unique adjoint kernel of L_θ, i.e.*

$$(L_\theta f, \sin\theta\cos^2\theta)_\theta = 0, \forall f \in L^2_\theta.$$

(the detailed calculation is as following:

$$\begin{aligned}
&\big(\partial_\theta^2 f - \partial_\theta(\tan\theta f) + 6f, \sin\theta\cos^2\theta\big)_{L^2_\theta} = \big(f, (\partial_\theta^2 + \tan\theta\partial_\theta + 6)(\sin\theta\cos^2\theta)\big)_{L^2_\theta} \\
&= \big(f, -7\sin\theta\cos^2\theta + 2\sin^3\theta + \tan\theta(\cos^3\theta - 2\sin^2\theta\cos\theta) + 6\sin\theta\cos^2\theta\big)_{L^2_\theta} = 0.
\end{aligned}$$

) Thus the condition (5.17) is necessary to eliminate some singularity in the elliptic estimates. Indeed, we have the following estimate if it does not hold:

$$\alpha^2\big\|D_z^2\Phi\big\|_{\mathcal{H}^k} + \left\|\partial_\theta^2\left(\Phi - \frac{1}{4\alpha}\sin 2\theta L_{12}F\right)\right\|_{\mathcal{H}^k} \leq C_k\|F\|_{\mathcal{H}^k}, \tag{5.19}$$

The proof is listed in the last of this section.

Remark 5.3. *It is noticeable that $\hat{\Phi}$ is linearly dependent on F. Indeed, we see that Φ is the image of F on the following operator:*

$$-L^{-1} - \frac{1}{4\alpha}\sin 2\theta L_{12}.$$

We'd like denote $\hat{\Phi}_f = -L^{-1}f - \frac{1}{4\alpha}\sin 2\theta L_{12}f$ in the further discussion. Particularly, $\hat{\Phi} = \Phi_F$ in our case.

Now we sketch the proof the Theorem 5.5. First the existence and uniqueness of the L^2-solution Φ comes from the standard L^p-theory as the orthogonal condition (5.17) holds?. Similarly with the proof of transport coercivity, we first establish a L^2- estimate (without weights), and apply it to derive a $\mathcal{H}^2-$elliptic estimate, after which the $\mathcal{H}^k-$case follows from a induction. During the proof, some angular Hardy-type estimates will be used(check them in the appendix **??**).

Lemma 5.2 (L^2-estimate). *Suppose Φ is the unique solution obtained above, then*

$$\left\|\partial_\theta \tilde{\Phi}\right\|_{L^2} + \left\|\partial_\theta^2 \Phi\right\|_{L^2} + \alpha^2 \left\|D_z^2 \Phi\right\|_{L^2} \leq 100\|F\|_{L^2}, \tag{5.20}$$

where $\tilde{\Phi} := \Phi/\cos\theta$.

Proof. **Step 1:** First we show that $\Phi_\star(z) = (\Phi(z,\theta), \sin\theta\cos^2\theta)_{L^2_\theta} \equiv 0$. Multiplying the both side of (5.15) with $\sin\theta\cos^2\theta$ and integrating in θ, we can see

$$\alpha^2 D_z^2 \Phi_\star + \alpha(5+\alpha) D_z \Phi_\star = (L_z\Phi, \sin\theta\cos^2\theta) = 0.$$

This is exactly a ODE with characteristic equation

$$\alpha^2\lambda(\lambda-1) + \alpha(5+\alpha)\lambda = 0 \Longrightarrow \lambda_1 = 0, \lambda = -5/\alpha.$$

Consequently, we have solution formulated as

$$\Phi_\star(z) = c_1 + c_2 z^{-5/\alpha}.$$

Moreover, the Dirichlet condition $\Phi(z,\theta) \to 0$ as $z \to \infty$ implies $c_1 = 0$, and $z^2\Phi|_{z=0} = 0$?implies $c_2 = 0$. In conclusion, we get $\Phi_\star(z) \equiv 0$.

Step 2: We derive the L^2-estimate for $\partial_\theta\Phi$. Multiplying the both side of (5.15) with Φ and integrating in (z,θ), it comes

$$\alpha^2\|D_z\Phi\|_{L^2}^2 - \alpha^2\|\Phi\|_{L^2}^2 + \frac{\alpha(5+\alpha)}{2}\|\Phi\|_{L^2}^2 + \|\partial_\theta\Phi\|_{L^2}^2 + \frac{1}{2}\|\Phi/\cos\theta\|_{L^2}^2 - 6\|\Phi\|_{L^2}^2 = (F,\Phi)_{L^2}.$$

(Just a bunch of integrations by part:

$$\begin{aligned}
\left(D_z^2\Phi, \Phi\right)_{L^2} &= \iint z^2\Phi\partial_z^2\Phi = -\iint (2z\Phi + z^2\partial_z\Phi)\partial_z\Phi \\
&= -\iint \left(z\partial_z(\Phi^2)\right) - \iint \left(z^2\partial_z^2\Phi\right) \\
&= -\left(\left\|D_z^2\Phi\right\|_{L^2}^2 - \left\|\Phi^2\right\|_{L^2}^2\right), \\
(D_z\Phi,\Phi) &= -\frac{1}{2}\|\Phi\|_{L^2}^2, \left(\partial_\theta^2\Phi,\Phi\right) = -\|\partial_\theta\Phi\|_{L^2}^2, \\
\left(\partial_\theta\left(\tan\theta\Phi\right),\Phi\right) &= -\iint \tan\theta\Phi\partial_\theta\Phi = -\frac{1}{2}\iint \tan\theta\partial_\theta(\Phi^2) = \frac{1}{2}\|\Phi/\cos\theta\|_{L^2}^2.
\end{aligned}$$

Notice the negative signs are shifted to the right side.) Consequently, since $-\alpha^2 + \alpha(5 + \alpha)/2 = \alpha(5 - \alpha) > 0$, we get

$$\|\partial_\theta \Phi\|_{L^2}^2 - 6\|\Phi\|_{L^2}^2 \le \|F\|_{L^2}\|\Phi\|_{L^2}. \tag{5.21}$$

Following we use the Fourier expansion of Φ:[13]

$$\Phi = \sum_{n\ge1} \Phi_n(z)\sin(2n\theta), \Phi_n(z) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \Phi(z,\theta)\sin(2n\theta)d\theta.$$

Then since $\{\sin 2n\theta, \cos 2n\theta\}$ is a family of orthogonal basis of $L^2([0,\pi/2])$ with norm $\|\sin 2n\theta\|_{L_\theta^2}^2 = \|\cos 2n\theta\|_{L_\theta^2}^2 = \pi/4$, we have

$$\begin{aligned}
\|\partial_\theta \Phi\|_{L^2}^2 =& \left\|\sum_{n\ge1} 2n\Phi_n(z)\cos 2n\theta\right\|_{L^2}^2 = \sum_{n\ge1} 4n^2\|\Phi_n\|_{L_z^2}^2\|\cos 2nx\|_{L^2}^2 = \frac{\pi}{4}\sum_{n\ge1} 4n^2\|\Phi_n\|_{L^2}^2,\\
-6\|\Phi\|_{L^2}^2 =& -6\left\|\sum_{n\ge1} \Phi_n(z)\sin 2n\theta\right\|_{L^2}^2 = -\frac{\pi}{4}\sum_{n\ge1} 6\|\Phi_n\|_{L^2}^2.
\end{aligned}$$

Since the coefficient is negative for $n = 1$, we'd like to write (5.21) as the following form:

$$\sum_{n\ge2}(4n^2 - 6)\|\Phi_n\|_{L_z^2}^2 \le 2\|\Phi_1\|_{L_z^2}^2 + \frac{4}{\pi}\|F\|_{L^2}\|\Phi\|_{L^2}. \tag{5.22}$$

To handle the term $\|\Phi_1\|_{L^2}^2$, we notice that[14]

$$\begin{aligned}
0 =& \Phi_\star(z) = \big(\Phi(z,\theta), \sin\theta\cos^2\theta\big)_{L_\theta^2} = \sum_{n\ge1}\Phi_n(z)\int_0^{\frac{\pi}{2}}(\sin\theta\cos^2\theta\sin 2n\theta)d\theta\\
=& \sum_{n\ge1}\frac{4n\cos(n\pi)}{16n^4 - 40n^2 + 9}\Phi_n = \sum_{n\ge1}(-1)^n\frac{4n}{(4n^2-9)(4n^2-1)}\Phi_n.
\end{aligned}$$

This implies

$$\begin{aligned}
\|\Phi_1\|_{L_z^2}^2 \le& \sum_{n\ge2}\left(\frac{15}{4}\frac{4n}{(4n^2-9)(4n^2-1)}\right)^2\|\Phi_n\|_{L_z^2}^2 = \sum_{n\ge2}\frac{225n^2}{(4n^2-9)^2(4n^2-1)^2}\|\Phi_n\|_{L_z^2}^2\\
\le& \sum_{n\ge2}\frac{225n^2}{(4n^2-9)^2(4n^2-1)^2}\|\Phi_n\|_{L_z^2}^2 \le \sum_{n\ge2}\frac{n^2}{(4n^2-9)^2}\|\Phi_n\|_{L_z^2}^2\\
\le& \sum_{n\ge2}\frac{1}{n^2}\|\Phi_n\|_{L_z^2}^2 \le \sum_{n\ge2}\|\Phi_n\|_{L^2}^2.
\end{aligned}$$

[13] After Fourier expansion we can transform the derivatives (to θ) into some algebraic operation and then handle them easily. The idea is similar with Fourier transform.

[14] Indefinite integral see Wolfram.

We substitute it into (5.22), then

$$\sum_{n\geq 2}\left(4n^2-8\right)\|\Phi_n\|_{L_z^2}^2\leq\frac{4}{\pi}\|F\|_{L^2}\|\Phi\|_{L^2}.$$

And then we have

$$\begin{aligned}\sum_{n\geq 1}(n^2+1)\|\Phi_n\|_{L_z^2}^2=&\sum_{n\geq 2}(n^2+1)\|\Phi_n\|_{L_z^2}^2+2\|\Phi_1\|_{L_z^2}^2\\ \leq&\sum_{n\geq 2}(n^2+3)\|\Phi_n\|_{L^2}^2\leq\sum_{n\geq 2}\left(4n^2-8\right)\|\Phi_n\|_{L_z^2}^2\\ \leq&\frac{4}{\pi}\|F\|_{L^2}\|\Phi\|_{L^2}^2\leq\frac{1}{4}\|F\|_{L^2}^2+\frac{4}{\pi}\|\Phi\|_{L^2}^2\\ \leq&\frac{1}{4}\|F\|_{L^2}^2+\sum_{n\geq 1}\|\Phi_n\|_{L_z^2}^2.\end{aligned}$$

Finally, we see

$$\|\partial_\theta\Phi\|_{L^2}^2=\frac{\pi}{4}\sum_{n\geq 1}4n^2\|\Phi_n\|_{L_z^2}^2\leq\frac{\pi}{4}\|F\|_{L^2}^2\Longrightarrow\|\partial_\theta\Phi\|_{L^2}\leq\|F\|_{L_z^2}.$$

Step 3: Using the bound for $\partial_\theta\Phi$ and Hardy-type inequalities, now we establish the estimate for $\partial_\theta^2\Phi, \partial_\theta(\Phi/\cos\theta)$ and $D_z^2\Phi$ respectively. First we test (5.15) with $\partial_\theta^2\Phi$ and get

$$\begin{aligned}\alpha^2\|D_z\partial_\theta\Phi\|_{L^2}^2-\alpha^2\|\partial_\theta\Phi\|_{L^2}^2+\frac{\alpha(5+\alpha)}{2}\|\partial_\theta\Phi\|_{L^2}^2+\left\|\partial_\theta^2\Phi\right\|_{L^2}^2&\\ -6\|\partial_\theta\Phi\|_{L^2}^2-\int_0^{\frac{\pi}{2}}\partial_\theta\left(\tan\theta\Phi\right)\partial_\theta^2\Phi=-\left(F,\partial_\theta^2\Phi\right)_{L^2}.&\end{aligned}$$

The main difficulty is the last term on the left side. Following we will show that it will be controlled by $\tilde{\Phi} = \Phi / \cos\theta$. Integrating by part to eliminate the high order term:

$$
\begin{aligned}
&-\int_0^{\frac{\pi}{2}} \partial_\theta (\tan\theta\Phi)\, \partial_\theta^2 \Phi \; d\theta = -\int_0^{\frac{\pi}{2}} \left(\sin\tilde{\Phi}\right) \partial_\theta^2 \left(\cos\theta\tilde{\Phi}\right) d\theta \\
&= -\int_0^{\frac{\pi}{2}} \left(\sin\theta\partial_\theta\tilde{\Phi} + \cos\theta\tilde{\Phi}\right)\left(-\cos\theta\tilde{\Phi} - 2\sin\theta\partial_\theta\tilde{\Phi} + \cos\theta\partial_\theta^2\tilde{\Phi}\right) \\
&= \int_0^{\frac{\pi}{2}} \left(2\sin^2\theta(\partial_\theta\tilde{\Phi})^2 - \sin\theta\cos\theta\partial_\theta\tilde{\Phi}\partial_\theta^2\tilde{\Phi} + \cos^2\theta\tilde{\Phi}^2 + 3\sin\theta\cos\theta\tilde{\Phi}\partial_\theta\tilde{\Phi} - \cos^2\theta\tilde{\Phi}\partial_\theta^2\tilde{\Phi}\right) d\theta \\
&= \int_0^{\frac{\pi}{2}} \Big(2\sin^2\theta(\partial_\theta\tilde{\Phi})^2 + \frac{1}{2}\partial_\theta(\sin\theta\cos\theta)(\partial_\theta\tilde{\Phi})^2 + \cos^2\theta\tilde{\Phi}^2 \\
&\qquad\quad + 3\sin\theta\cos\theta\tilde{\Phi}\partial_\theta\tilde{\Phi} + \cos^2\theta(\partial_\theta\tilde{\Phi})^2 - 2\sin\theta\cos\theta\tilde{\Phi}\partial_\theta\tilde{\Phi}\Big) d\theta \\
&= \int_0^{\frac{\pi}{2}} \Big(2\sin^2\theta(\partial_\theta\tilde{\Phi})^2 + \frac{1}{2}(\cos^2\theta - \sin^2\theta)(\partial_\theta\tilde{\Phi})^2 + \cos^2\theta\tilde{\Phi}^2 \\
&\qquad\quad - \frac{1}{2}(\cos^2\theta - \sin^2\theta)\tilde{\Phi}^2 + \cos^2\theta(\partial_\theta\tilde{\Phi})^2\Big) d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{3}{2}(\partial_\theta\tilde{\Phi})^2 + \frac{1}{2}\tilde{\Phi}^2\right) d\theta.
\end{aligned}
$$

Substitute this into the test equation and neglect the radial terms(as them keep positive), then we finally get:

$$
\begin{aligned}
\left\|\partial_\theta^2\Phi\right\|_{L^2}^2 + \frac{3}{2}\left\|\partial_\theta\tilde{\Phi}\right\|_{L^2}^2 \leq & 6\|\partial_\theta\Phi\|_{L^2}^2 + \frac{1}{2}\left\|\tilde{\Phi}\right\|_{L^2}^2 - \left(F, \partial_\theta^2\Phi\right)_{L^2} \\
\leq & 11\|\partial_\theta\Phi\|_{L^2}^2 + 5\|\partial_\theta\Phi\|_{L^2}^2 + \frac{1}{2}\left\|\partial_\theta^2\Phi\right\|,
\end{aligned}
$$

which yields

$$
\left\|\partial_\theta^2\Phi\right\|_{L^2}^2 + 3\left\|\partial_\theta\tilde{\Phi}\right\|_{L^2}^2 \leq 23\|F\|_{L^2}^2.
$$

The radial part can be obtained by a similar argument▲. □

Next we give out the proof of $k = 2$ case for Theorem 5.5.

Proof of Theorem 5.5. We will add it later. □

Proof of Remark 5.2. Recall that $\sin 2\theta$ is in the kernel of $L_\theta = \partial_\theta^2 - \partial_\theta(\tan\theta\cdot) + 6\,\mathrm{Id}$ since

$$
\partial_\theta^2(\sin 2\theta) - \partial_\theta(\tan\theta\sin 2\theta) + 6\sin 2\theta = -4\sin 2\theta - 2\sin 2\theta + 6\sin 2\theta = 0.
$$

Consequently, we consider $\hat{\Phi} = \Phi - g(z)\sin 2\theta$ such that

$$
L\hat{\phi} = L\Phi - L(g\sin 2\theta) = F - L_z g\sin 2\theta =: \tilde{F}.
$$

Now it remains to determine g such that $\tilde{F}_\star = (\tilde{F}, \sin\theta\cos^2\theta) \equiv 0$, and then the elliptic estimate holds for $\hat{\Phi}$ immediately. Accordingly, g is determined by

$$\left(F - L_z g \sin 2\theta, \sin\theta\cos^2\theta\right)_{L^2_\theta} \equiv 0,$$

which yields:

$$L_z g = \left(\int_0^{\frac{\pi}{2}} \sin 2\theta \sin\theta \cos^2\theta d\theta\right)^{-1} \left(F, \sin\theta\cos^2\theta\right)_\theta = \frac{15}{4} F_\star.$$

This is a linear ODE and we can solve it via integral factor:

$$\begin{aligned} g(z) =& -\frac{15}{4\alpha^2}\int_z^\infty \rho^{-(\frac{5}{\alpha}+1)} \int_0^\rho s^{\frac{5}{\alpha}-1} F_\star(s) ds d\rho \\ =& \frac{3}{4\alpha}\int_z^\infty \partial_\rho(\rho^{-\frac{5}{\alpha}}) \int_0^\rho s^{\frac{5}{\alpha}-1} F_\star(s) ds d\rho \\ =& -\frac{3}{4\alpha} z^{-\frac{5}{\alpha}} \int_0^z \rho^{\frac{5}{\alpha}-1} F_\star(\rho) d\rho - \frac{3}{4\alpha}\int_z^\infty \frac{F_\star(\rho)}{\rho} d\rho \\ =&: \bar{g} - \frac{1}{4\alpha}\int_z^\infty \int_0^{\frac{\pi}{2}} \frac{F \cdot 3\sin\theta\cos^2\theta}{\rho} d\theta d\rho. \end{aligned}$$

We can easily prove that $\bar{g}$ is of low order with estimate $\|\bar{g}\|_{L^2} \le C\|F\|_{L^2}$. And the later term

$$L_{12}F := \int_z^\infty \int_0^{\frac{\pi}{2}} \frac{F(z', \theta')K(\theta)}{z'} d\theta' dz', K(\theta) = 3\sin\theta\cos^2\theta,$$

is the main singularity. □

5.6 A priori estimate

In our final calculation we aim to get a priori estimate for g satisfies $L_{12}g(0) = 0$. We recall that

$$\mathcal{L}_\Gamma^T g = \mathbb{P}(\mathcal{N}_0 + \mathcal{N} + \mathcal{N}_*) = \mathcal{N}_0 + \mathcal{N} + \mathcal{N}_* - L_{12}(\mathcal{N}_0 + \mathcal{N})(0)\frac{\Gamma}{c^*}\frac{2z^2}{(1+z)^3}.$$

And consequently,

$$\left(\mathcal{L}_\Gamma^T g, g\right)_{\mathcal{H}^4} \le |(\mathcal{N}_0, g)_{\mathcal{H}^4}| + |(\mathcal{N}, g)_{\mathcal{H}^4}| + |(\mathcal{N}_*, g)_{\mathcal{H}^4}| + |L_{12}(\mathcal{N}_0 + \mathcal{N})(0)| \left\|\frac{\Gamma}{c^*}\frac{2z^2}{(1+z)^3}\right\|_{\mathcal{H}^4} \|g\|_{\mathcal{H}^4}.$$

We will finish the following estimate respectively:

$$\begin{aligned} |(\mathcal{N}_0, g)_{\mathcal{H}^4}| &\le C(\alpha^2\|g\|_{\mathcal{H}^4} + \alpha^{\frac{1}{2}}\|g\|^2_{\mathcal{H}^4} + \alpha^{-\frac{3}{2}}\|g\|^3_{\mathcal{H}^4}), \\ |(\mathcal{N}, g)_{\mathcal{H}^4}| &\le C(\alpha^2\|g\|_{\mathcal{H}^4} + \alpha^{\frac{1}{2}}\|g\|^2_{\mathcal{H}^4} + \alpha^{-\frac{3}{2}}\|g\|^3_{\mathcal{H}^4}), \\ |\mu| &\le C\left(\alpha + \alpha^{-\frac{3}{2}}\|g\|_{\mathcal{H}^4} + \alpha^{-\frac{5}{2}}\|g\|^2_{\mathcal{H}^4}\right), \\ |(\mathcal{N}_*, g)_{\mathcal{H}^4}| \le C|\mu|\|g\|^2_{\mathcal{H}^4} &\le C\left(\alpha\|g\|^2_{\mathcal{H}^4} + \alpha^{-\frac{3}{2}}\|g\|_{\mathcal{H}^4}\|g\|^3_{\mathcal{H}^4} + \alpha^{-\frac{5}{2}}\|g\|^4_{\mathcal{H}^4}\right). \end{aligned}$$

And in conclusion,

$$\|g\|_{\mathcal{H}^4}^2 \leq (\mathcal{L}_\Gamma^T g, g)_{\mathcal{H}^4} \leq C \left(\alpha^2 \|g\|_{\mathcal{H}^4} + \alpha^{\frac{1}{2}} \|g\|_{\mathcal{H}^4}^2 + \alpha^{-\frac{3}{2}} \|g\|_{\mathcal{H}^4} \|g\|_{\mathcal{H}^4}^3 + \alpha^{-\frac{5}{2}} \|g\|_{\mathcal{H}^4}^4 \right).$$

Particularly, we shall see that $\|g\|_{\mathcal{H}^4} \leq C\alpha^{\frac{7}{4}} \Longrightarrow \|g\|_{\mathcal{H}^4} \leq C\alpha^2$.

5.7 Appendix

Projector